

Pushout of quasi-finite and flat group schemes over a Dedekind ring

Marco Antei*

Abstract. Let G , G_1 and G_2 be quasi-finite and flat group schemes over a complete discrete valuation ring R , $\varphi_1 : G \rightarrow G_1$ any morphism of R -group schemes and $\varphi_2 : G \rightarrow G_2$ a model map. We construct the pushout P of G_1 and G_2 over G in the category of R -affine group schemes. In particular when φ_1 is a model map too we show that P is still a model of the generic fibre of G . We also provide a short proof for the existence of cokernels and quotients of finite and flat group schemes over any Dedekind ring.

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1 Introduction

1.1 Aim and scope

We are interested in the construction of the pushout (whose definition will be recalled in §2) in the category of affine group schemes over a given ring as

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described hereafter. It is known that in the category of abstract groups the pushout of two groups over a third one always exists but it is not finite even when the three groups are all finite (unless one takes very particular cases). However for group schemes over a Dedekind ring R something new happens when we consider some special important cases: so let G , G_1 and G_2 be R -affine group schemes and consider the diagram

$$\begin{array}{ccc} & G & \\ \varphi_1 \swarrow & & \searrow \varphi_2 \\ G_1 & & G_2 \end{array} \quad (1)$$

where $\varphi_i : G \rightarrow G_i$ ($i = 1, 2$) are R -group scheme morphisms. We first prove the following

Theorem 1.1. *(Cf. Theorem 3.2) Assume R is a complete discrete valuation ring and G , G_1 , G_2 are finite and flat over R . Then if φ_1 is a model map (i.e. generically an isomorphism) the pushout of (1) in the category of affine R -group schemes exists. Moreover it is finite and flat and its generic fibre is isomorphic to $G_{2,K}$, the generic fibre of G_2 .*

This immediately implies that when G , G_1 and G_2 are all models of a same K -group scheme G_K (K being the fraction field of R) then the pushout of (1) exists and is still a model of G_K thus proving the existence of a lower bound for models of finite group schemes. This was already known in the commutative case (cf. [9], Proposition 2.2.2). The same will be true for the quasi-finite case under the assumption that $G_{2,K}$ admits a finite and flat R -model:

Theorem 1.2. *(Cf. Theorem 3.5) Assume R is a complete discrete valuation ring and G , G_1 , G_2 are quasi-finite and flat over R . If φ_1 is a model map and $G_{2,K}$ admits a finite and flat model then the pushout of (1) in the category of affine R -group schemes exists. Moreover it is quasi-finite and flat and its generic fibre is isomorphic to $G_{2,K}$.*

Using the fact that $G_{2,K}$ always admits, when it is étale, a finite and flat model up to a finite extension of scalars we finally prove the following

Corollary 1.3. *(Cf. Corollary 3.9) Assume R is a complete discrete valuation ring and G , G_1 , G_2 are quasi-finite and flat over R . Then if φ_1 is a model map and $G_{2,K}$ is étale then the pushout of (1) in the category of affine R -group schemes exists. Again it is quasi-finite and flat and its generic fibre is isomorphic to $G_{2,K}$.*

All the proofs rest on the computation of the pushout in the category of R -Hopf algebras. With the same techniques we briefly study in section §3.3 the existence of cokernels in the category of affine R -group schemes where R is any Dedekind ring. This will lead to a new and short proof of the the following:

Corollary 1.4. *(Cf. Corollary 3.13) Let R be a Dedekind ring, G and H two finite and flat R -group schemes with H a closed and normal R -subgroup scheme of G . Then the quotient G/H exists in the category of R -affine group schemes.*

This holds over any base scheme and is in fact a consequence of a much bigger theorem (Cf. [4] Théorème 7.1).

1.2 Notations and conventions

Every ring A will be supposed to be associative and unitary, i.e. provided with a unity element denoted by 1_A , or simply 1 if no confusion can arise. However, unless stated otherwise, a ring will not be supposed to be commutative. Every Dedekind ring, instead, will always be supposed to be commutative. For an R -algebra A the morphisms $u_A : R \rightarrow A$ and $m_A : A \otimes_R A \rightarrow A$ will always denote the unity and the multiplication morphisms (respectively). If moreover A has an R -coalgebra structure then $\Delta_A : A \rightarrow A \otimes_R A$, $\varepsilon_A : A \rightarrow R$ will denote the comultiplication and the counity respectively. Furthermore if A has a R -Hopf algebra structure then $S_A : A \rightarrow A$ will denote the coinverse. All the coalgebra structures will be supposed to be coassociative. Morphisms of R -algebras (resp. R -coalgebras, R -Hopf algebras) are R -module morphisms preserving R -algebra (resp. R -coalgebra, R -Hopf algebra) structure. We denote by $R\text{-Hopf}$ the category of associative and coassociative R -Hopf algebras while $R\text{-Hopf}_{ff}$ will denote the category of associative and coassociative R -Hopf algebras which are finite and flat as R -modules. When $R \rightarrow T$ is a morphism of commutative algebras, M is a R -module, X is a R -scheme, $f : M \rightarrow N$ is a R -module morphism and $\varphi : X \rightarrow Y$ a morphism of R -schemes then we denote by M_T , X_T , $f_T : M_T \rightarrow N_T$ and $\varphi_T : X_T \rightarrow Y_T$ respectively the T -module $M \otimes_R T$, the T -scheme $X \times_{\text{Spec}(R)} \text{Spec}(T)$, the T -module morphism induced by f and the T -morphism of schemes induced by φ . When R is a Dedekind ring and K its field of fractions then a R -morphism of schemes $\varphi : X \rightarrow Y$ is called a model map if generically it is an isomorphism, i.e. $\varphi_K : X_K \rightarrow Y_K$ is an isomorphism.

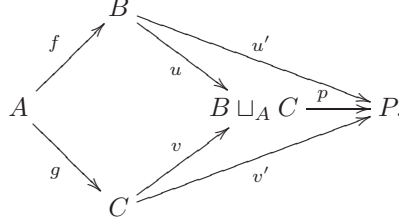
2 Pushout of Hopf algebras

In this section we first study the pushout of algebras over a commutative ring R then we discuss the existence of the pushout in the category of $R\text{-Hopf}_{ff}$ when R is a complete discrete valuation ring. Let us first recall that in a category \mathcal{C} the pushout (see for instance [8], III, §3) of a diagram

$$\begin{array}{ccc} & A & \\ f \swarrow & & \searrow g \\ B & & C \end{array} \quad (2)$$

(where clearly A, B, C are objects of \mathcal{C} and f, g morphisms in the same category) is an object of \mathcal{C} that we denote $B \sqcup_A C$ provided with two morphisms $u : B \rightarrow$

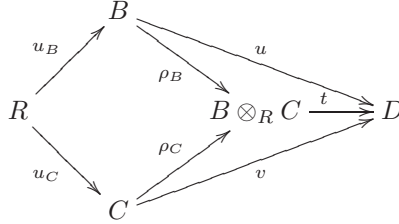
$B \sqcup_A C$, $v : C \rightarrow B \sqcup_A C$ such that $uf = vg$ and satisfying the following universal property: for any object P of \mathcal{C} and any two morphisms $u' : B \rightarrow P$, $v' : C \rightarrow P$ in \mathcal{C} such that $u'f = v'g$ then there exists a unique morphism $p : B \sqcup_A C \rightarrow P$ making the following diagram commute:



When A is an initial object (provided it exists) of \mathcal{C} then $B \sqcup_A C$ is the coproduct¹ of B and C in \mathcal{C} . When \mathcal{C} is the category of commutative R -algebras then the pushout is given by the tensor product $B \otimes_A C$. This is not true anymore if \mathcal{C} is the category of R -algebras, (cf. Example 2.9 or create easier examples). However we can always find a pushout even when \mathcal{C} is the category of R -algebras and it will be denoted by $B *_A C$. Before introducing, however, the pushout for non (necessarily) commutative R -algebras we recall the behavior of the tensor product over R . We put ourselves in the following situation:

Notation 2.1. By R we will denote a commutative ring while A , B and C will be R -algebras and $f : A \rightarrow B$, $g : A \rightarrow C$ two R -algebra morphisms. We also denote by $\rho_B : B \rightarrow B \otimes_R C$ and $\rho_C : C \rightarrow B \otimes_R C$ the morphisms sending respectively $b \mapsto b \otimes 1_C$ and $c \mapsto 1_B \otimes c$.

Proposition 2.2. Let D be any R -algebra and $u : B \rightarrow D$, $v : C \rightarrow D$ two R -algebra morphisms such that $u \circ u_B = v \circ u_C$ and such that $u(b)v(c) = v(c)u(b)$ for all $b \in B, c \in C$. Then there exists a unique R -algebra morphism $t : B \otimes_R C \rightarrow D$ making the following diagram commute:



Proof. Cf. for instance [3], I, §3 Proposition 3.2. □

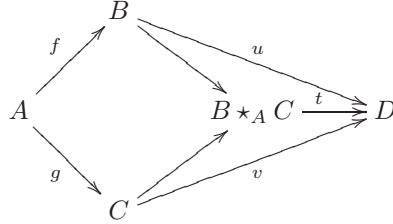
Unfortunately $B \otimes_A C$ behaves badly in general and one can observe that even $A \otimes_A A \simeq A$, as an R -algebra, is not a natural quotient of $A \otimes_R A$. So, instead, let us consider the following construction:

¹The coproduct can be defined, however, without assuming the existence of an initial object.

Definition 2.3. We denote by $B \star_A C$, and we call it the star product of B and C over R , the quotient of $B \otimes_R C$ by the two-sided ideal generated by A , i.e. the ideal of $B \otimes_R C$ generated by the set $\{\rho_B f(a) - \rho_C g(a)\}_{a \in A}$.

It is an easy consequence the following universal property of the star product:

Proposition 2.4. *Let D be any R -algebra and $u : B \rightarrow D$, $v : C \rightarrow D$ two R -algebra morphisms such that $uf = vg$ and such that $u(b)v(c) = v(c)u(b)$ for all $b \in B, c \in C$. Then there exists a unique R -algebra morphism $t : B \star_A C \rightarrow D$ making the following diagram commute:*



Proof. It is sufficient to take the R -algebra morphism $B \otimes_R C \rightarrow D$ and observe that it passes to the quotient. \square

The star product will only be used in Example 2.8 and §3.3, so finally let us recall the construction of the pushout of R -algebras: we follow essentially [2] 1.7 and 5.1 with very few modifications in the exposition in order to obtain an easier to handle description. We describe A , B and C giving their presentation as R -algebras thus getting $R\langle X_0; S_0 \rangle$, $R\langle X_1; S_1 \rangle$ and $R\langle X_2; S_2 \rangle$ respectively, where X_i is a generating set with relations S_i ($i = 0, 1, 2$). We recall that $R\langle X; S \rangle$ is to be intended as the R -algebra whose elements are all R -linear combinations of words on the set X quotiented by the two-sided ideal generated by the relations in S . Observe that for $y, z \in X$ we are not assuming $zy = yz$; if it is the case the information will appear in S . However for any $x \in X$ and any $r \in R$ we do assume $xr = rx$. For example the commutative R -algebra $R[x, y]/f(x, y)$ can be presented as $R\langle x, y; f(x, y) = 0, xy = yx \rangle$. First we observe that the coproduct of B and C (i.e. the pushout of B and C over the initial object R) is given by the R -algebra $B *_R C := R\langle X_1 \cup X_2; S_1 \cup S_2 \rangle$ where the union is of course disjoint. Let us denote by $u : B \rightarrow B *_R C$ and $v : C \rightarrow B *_R C$ the canonical inclusions. Then the pushout of B and C over A is given by the R -algebra

$$B \star_A C := R\langle X_1 \cup X_2; S_1 \cup S_2 \cup S_3 \rangle \quad (3)$$

where S_3 consists on the relations given by $uf(x) = vg(x)$ for every $x \in X_0$. Now we relate the pushout just described to the tensor product:

Lemma 2.5. *Assume that $B = R\langle X_1; S_1 \rangle$ and $C = R\langle X_2; S_2 \rangle$. Then $B \otimes_R C$ can be presented as $R\langle X_1 \cup X_2; S_1 \cup S_2, \{zy = yz\}_{z \in X_1, y \in X_2} \rangle$ thus becoming a quotient of $R\langle X_1 \cup X_2; S_1 \cup S_2 \rangle = B *_R C$.*

Proof. Let D be an R -algebra provided with R -algebra morphisms $m : B \rightarrow D$ and $n : C \rightarrow D$ such that $m \circ u_B = n \circ u_C$, and assume moreover that $m(b)n(c) = n(c)m(b)$ for all $b \in B, c \in C$. Let us denote by $u : B \rightarrow B *_R C$ and $v : C \rightarrow B *_R C$ the canonical morphisms and by $\lambda : B *_R C \rightarrow D$ the universal morphism making the following diagram commute:

$$\begin{array}{ccccc}
 & & B & & \\
 & u_B \nearrow & & \searrow m & \\
 R & & & & D \\
 & u_C \searrow & & \nearrow \lambda & \\
 & & B *_R C & & \\
 & v \nearrow & & \searrow n & \\
 & & C & &
 \end{array}$$

By assumption $\lambda u(z)\lambda v(y) = \lambda v(y)\lambda u(z)$ so $u(z)v(y) - v(y)u(z) \in \ker(\lambda)$ hence λ factors through $R\langle X_1 \cup X_2; S_1 \cup S_2, \{zy = yz\}_{z \in X_1, y \in X_2} \rangle$ providing it with the universal property stated in Proposition 2.2 and this is enough to conclude. \square

Let $q : R \rightarrow T$ be a R -commutative algebra. When $f = f(x_1, \dots, x_n) \in R[x_1, \dots, x_n]$ we denote by $q_*(f)$ the polynomial in $T[x_1, \dots, x_n]$ whose coefficients are the image in T by q of the coefficients of f , i.e. the image of f through the morphism $q_* : R[x_1, \dots, x_n] \rightarrow T[x_1, \dots, x_n] = R[x_1, \dots, x_n] \otimes_R T$. Now take $R\langle X, S \rangle$: by an abuse of notation we denote by $q_*(S)$ the set of relations $\{q_*(s_i) = 0\}$ on the set X . In Lemma 2.6 we observe that the pushout is stable under base change.

Lemma 2.6. *Let $q : R \rightarrow T$ be a R -commutative algebra and $R\langle X; S \rangle$ any R -algebra, then*

1. $R\langle X; S \rangle \otimes_R T \simeq T\langle X; q_*(S) \rangle$,
2. $(B *_A C) \otimes_R T \simeq (B \otimes_R T) *_{(A \otimes_R T)} (C \otimes_R T)$.

Proof. As a commutative R -algebra, T is isomorphic to $R[\{y_i\}]/(\{f_r\})$ where $\{y_i\}$ is a set of generators and $\{f_r\}$ a set of polynomials in the variables $\{y_i\}$ with coefficients in R . So by Lemma 2.5 $R\langle X; S \rangle \otimes_R T$ is isomorphic to $R\langle X \cup \{y_i\}; S \cup \{f_r = 0\} \cup \{y_i y_j = y_j y_i\} \cup \{x y_i = y_i x\}_{x \in X} \rangle$ which is isomorphic to $R\langle X \cup \{y_i\}; q_*(S) \cup \{f_r = 0\} \cup \{y_i y_j = y_j y_i\} \cup \{x y_i = y_i x\}_{x \in X} \rangle$ and the latter is isomorphic to $T\langle X; q_*(S) \rangle$ since T commutes with X and this proves 1. Let us describe A , B and C as $R\langle X_0; S_0 \rangle$, $R\langle X_1; S_1 \rangle$ and $R\langle X_2; S_2 \rangle$ respectively. As a consequence of point 1 we have $A \otimes_R T \simeq T\langle X_0; q_*(S_0) \rangle$, $B \otimes_R T \simeq T\langle X_1; q_*(S_1) \rangle$, $C \otimes_R T \simeq T\langle X_2; q_*(S_2) \rangle$ and $(B *_A C) \otimes_R T \simeq T\langle X_1 \cup X_2; q_*(S_1 \cup S_2 \cup S_3) \rangle$ where S_3 is as described in (3). But $(B \otimes_R T) *_{(A \otimes_R T)} (C \otimes_R T)$ is also isomorphic to the latter which enables us to conclude. \square

Notation 2.7. When R is a Dedekind ring and M an R -module, let us denote by $q : M \rightarrow F(M)$ the unique quotient (cf. [5], Lemme (2.8.1.1)) of M which is R -flat and such the induced map $q_K : M_K \rightarrow F(M)_K$ is an isomorphism.

Let us analyze a few examples whose importance will be clear in the following sections:

Example 2.8. Let $f : A \rightarrow B$ and $g : A \rightarrow R$ be morphisms of R -algebras then the canonical morphism $\varphi : B *_A R \rightarrow B *_A R$ is an isomorphism. Indeed we observe that $B *_R R = B$ and that the canonical morphisms $u : B \rightarrow B *_R R$ and $v : R \rightarrow B *_R R$ are nothing else but Id_B and the unit morphism u_B respectively, then for any $b \in B$ and any $r \in R$ we have $u(b)v(r) = v(r)u(b)$. Hence denoting by $u' : B \rightarrow B *_A R$ and $v' : R \rightarrow B *_A R$ the canonical morphisms we also have $u'(b)v'(r) = v'(r)u'(b)$ as $u' = \lambda u$ and $v' = \lambda v$ where $\lambda : B *_R R \rightarrow B *_A R$ is the universal morphism. Then φ can be inverted according to Proposition 2.4. Observe that $B *_A R$ is finite as an R -module if B is finite (it is indeed a quotient of B).

Example 2.9. Let R be a discrete valuation ring with uniformising element π . Let us fix a positive integer p and let us set $A := R[x]/x^p$, $B := R[y]/y^p$ and $C := R[z]/z^p$ (thus commutative R -algebras). Consider the morphisms $f : A \rightarrow B, x \mapsto \pi^n y$ and $g : A \rightarrow C, x \mapsto \pi^m z$ where $m > n > 0$ are integers. Then $B *_A C = R\langle y, z; y^p = z^p = 0, \pi^m z = \pi^n y \rangle$. Observe that, as an R -module, $B *_A C$ is not flat as $\pi^n(\pi^{m-n}z - y) = 0$ thus $\pi^{m-n}z - y$ is a R -torsion element. However if we add the relation $\pi^{m-n}z = y$ then we eliminate torsion from $B *_A C$ and what we obtain is (cf. Notation 2.7) $F(B *_A C) = R\langle y, z; y^p = z^p = 0, \pi^{m-n}z = y \rangle = R[z]/z^p$ thus finitely generated and flat and, in this particular case, it is isomorphic to $F(B \otimes_A C)$.

The following well known result will be used several times in this paper:

Theorem 2.10. *Let R be a complete discrete valuation ring with fraction field K and residue field k . Let M be a torsion-free R -module of finite rank r (i.e. $r := \dim_K(M \otimes_R K) < +\infty$). Then $M \simeq_{R\text{-mod}} K^{\oplus r-s} \oplus R^{\oplus s}$, where $s = \dim_k(M \otimes_R k)$.*

Proof. This is [6], Chapter 16, Corollary 2, □

Theorem 2.10 is not true when R is not complete (cf. [6], Theorem 19) and this is why we will often need to restrict to complete discrete valuation rings. The following lemma is crucial in this paper:

Lemma 2.11. *Let R be a complete discrete valuation ring and assume that $f : A \rightarrow B$ and $g : A \rightarrow C$ are R -algebra morphisms where furthermore $g_K : A_K \rightarrow C_K$ is an isomorphism. Then if A, B and C are finitely generated and flat as R -modules then the same holds for $F(B *_A C)$. Moreover the canonical R -algebra morphism $B \rightarrow F(B *_A C)$ induces an isomorphism $B_K \rightarrow F(B *_A C)_K$.*

Proof. Let π be an uniformising element of R and let K and k be the fraction and residue fields respectively. As usual let us present by $R\langle X_0; S_0 \rangle$, $R\langle X_1; S_1 \rangle$ and $R\langle X_2; S_2 \rangle$ respectively the R -algebras A, B and C where for X_0, X_1, X_2 we take respectively bases of A, B, C as R -modules minus the identity elements so that the cardinality x_0, x_1, x_2 of those sets is the rank of A, B and C (respectively)

minus one; of course $x_0 = x_2$. Now $B *_A C = R\langle X_1 \cup X_2; S_1 \cup S_2 \cup S_3 \rangle$ where S_3 is as described in (3). In particular S_3 is a set made of x_0 R -linear relations relating the x_2 elements of X_2 and the x_1 elements of X_1 . As in Example 2.9 the information of being R -torsion (if any) is contained in the set S_3 , so if we want to cut out R -torsion we need to add another set of x_0 relations S_4 obtained as follows: for each relation $(s = 0) \in S_3$ add the relation $(t = 0)$ to S_4 where $s = \pi^v t$ and t has at least one coefficient equal to an invertible element of R . Thus $F(B *_A C) = R\langle X_1 \cup X_2; S_1 \cup S_2 \cup S_3 \cup S_4 \rangle$. But since relations in S_3 are automatically satisfied if we add S_4 then $F(B *_A C) = R\langle X_1 \cup X_2; S_1 \cup S_2 \cup S_4 \rangle$. Now Lemma 2.6, point 1, implies that $F(B *_A C) \otimes_R k = k\langle X_1 \cup X_2; q_*(S_1 \cup S_2 \cup S_4) \rangle$ where $q : R \twoheadrightarrow k$ is the canonical surjection so $F(B *_A C) \otimes_R k$ is the quotient of $k\langle X_1 \cup X_2; q_*(S_1 \cup S_2) \rangle$ by the two-sided ideal generated by the relations $q_*(S_4)$. But in $k\langle X_1 \cup X_2; q_*(S_1 \cup S_2) \rangle$ the elements of the set $X_1 \cup X_2$ are $x_1 + x_2$ k -linearly independent vectors then if we add the $x_0 = x_2$ k -linear relations $q_*(S_4)$ what remains is a set of at least $x_1 = rk(B) - 1$ k -linearly independent elements which become $rk(B)$ if we add 1_B . Combining this with Theorem 2.10 we obtain that $F(B *_A C)$ is a finitely generated R -free module, as required, as $\dim_k(F(B *_A C)) \otimes_R k = \dim_K(F(B *_A C)) \otimes_R K$. The last assertion follows easily from Lemma 2.6, point 2. \square

Remark 2.12. The construction in Lemma 2.11 does not depend on A . That means that if we take A' , $f' : A' \rightarrow B$ and $g' : A' \rightarrow C$ satisfying similar assumptions then $F(B *_A C) \simeq F(B *_A' C)$. Indeed, again by [5], Lemme 2.8.1.1, we observe that $F(B *_A C)$ is isomorphic to the unique quotient of $B *_R C$ which is R -flat and whose tensor over K gives B_K ; but the same property is satisfied by $F(B *_A' C)$ hence we conclude by unicity.

Proposition 2.13. *Let R be any commutative ring. Then the pushout in the category of R -bialgebras exists.*

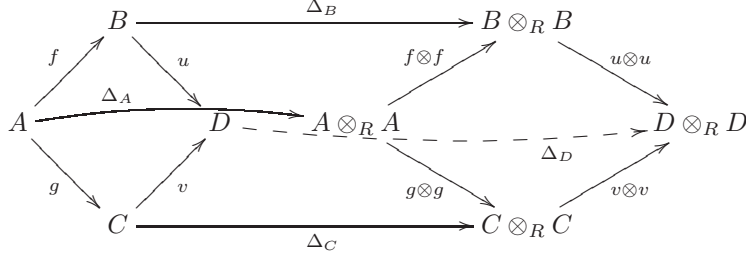
Proof. We follow² [7], Chapitre 5, §5.1, Proposition. Consider the diagram

$$\begin{array}{ccc}
 & & B \\
 & \nearrow u_B & \\
 R & \xrightarrow{u_A} & A \\
 & \searrow u_C & \\
 & & C
 \end{array}
 \begin{array}{l}
 \\
 \\
 \\
 \end{array}
 \begin{array}{l}
 \\
 f \\
 \\
 g
 \end{array}
 \tag{4}$$

where we assume that A , B and C are R -bialgebras and the arrows are R -Hopf algebra morphisms. Let $D := B *_A C$ be the pushout of the diagram in the category of R -algebras and let m_D and u_D be respectively the multiplication and the unit morphism. Then we need to provide D with a comultiplication Δ_D and a counit ε_D such that $(D, m_D, u_D, \Delta_D, \varepsilon_D)$ is a R -bialgebra. We describe how

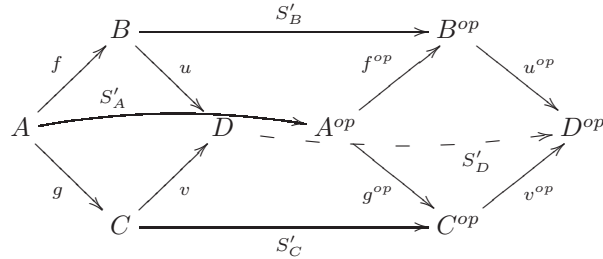
²In [7], however, Lemaire uses different notations.

to construct Δ_D , the construction of ε_D being easier. The rest will be standard verification over complicated diagrams. The existence of Δ_D is explained in the following diagram, taking into account the universal property of D :



□

Remark 2.14. Notation being as in Proposition 2.13 one observes that we can define a R -module morphism $S_D : D \rightarrow D$, candidate to be a coinverse, as follows: first construct the opposite algebras A^{op} , B^{op} , C^{op} , D^{op} , the opposite morphisms f^{op} , g^{op} , u^{op} , v^{op} and the morphisms of R -algebras S'_A , S'_B , S'_C , induced by the R -algebra anti-morphisms S_A , S_B , S_C . Then the existence of S'_D follows from the following diagram



exploiting the universal property of D then S_D is the anti-morphism induced by S'_D . However S_D may fail to be a coinverse for D as $m_D \circ (S_D \otimes id_D) \circ m_D$ may not be equal to $u_D \circ \varepsilon_D$ (same for $m_D \circ (id_D \otimes S_D) \circ m_D$).

In order to have an explicit description for S_D , constructed in Remark 2.14, set, as usual, $A = R\langle X_0; S_0 \rangle$, $A = R\langle X_1; S_2 \rangle$ and $C = R\langle X_2; S_2 \rangle$ so $D = R\langle X_1 \cup X_2; S_1 \cup S_2 \cup S_3 \rangle$ where S_3 is as described in (3); it is sufficient to set $S_D(x_1) := S_B(x_1)$ for any $x_1 \in X_1$, $S_D(x_2) := S_C(x_2)$ for any $x_2 \in X_2$ for any $x_1 \in X_1$, $S_D(x_1 x_2) := S_D(x_2) S_D(x_1)$ and $S_D(x_2 x_1) := S_D(x_1) S_D(x_2)$. It is well defined and is by construction an anti-isomorphism for the R -algebra D . A similar construction gives an explicit description of Δ_D , taking into account that Δ_D is a morphism of R -algebras and not an anti-morphism.

Corollary 2.15. *Let R be a complete discrete valuation ring and assume that $f : A \rightarrow B$ and $g : A \rightarrow C$ are R -algebra morphisms where furthermore $g_K : A_K \rightarrow C_K$ is an isomorphism. Then $F(B *_A C)$ has a natural structure of*

*R-Hopf algebra. If moreover A , B and C are finitely generated and flat as R -modules then $F(B *_A C)$ is the pushout of B and C over A in $R\text{-Hopf}_{ff}$.*

Proof. By [5], (2.8.3) and of course Proposition 2.13 we obtain that $F(B *_A C)$ has a natural structure of R -bialgebra. We need to prove the existence of a coinverse $S_{F(B *_A C)}$ that gives $F(B *_A C)$ a natural structure of R -Hopf algebra. So let us take for $D := B *_A C$ the R -module morphism S_D defined in Remark 2.14. This morphism induces (by [5], Lemme 2.8.3) an R -module morphism $S_{F(D)} : F(D) \rightarrow F(D)$ which is the required coinverse: indeed

$$m_{F(D)} \circ (S_{F(D)} \otimes id_{F(D)}) \circ m_{F(D)} - u_D \circ \Delta_D \quad (5)$$

is the zero map 0_D and this is clear since $F(D) \subset B_K$ and (5) tensored over K gives rise to the equality

$$m_{B_K} \circ (S_{B_K} \otimes id_{B_K}) \circ m_{B_K} = u_{B_K} \circ \Delta_{B_K}$$

which holds as B_K is a K -Hopf algebra. The same is true for $m_{F(D)} \circ (id_{F(D)} \otimes S_{F(D)}) \circ m_{F(D)}$. Finally $F(B *_A C)$ is finitely generated and flat as an R -module when A , B and C are: this is Lemma 2.11. \square

Remark 2.16. Notation being as in Proposition 2.13, we observe that $B *_A C$ is cocommutative if A , B , C are. The same conclusion holds, then, for $F(B *_A C)$ in Corollary 2.15. Moreover observe that if A , B , C are commutative then $F(B *_A C)$ is commutative too since it is contained in B_K . So in particular in this case $F(B *_A C) \simeq F(B \otimes_A C)$, as it happend in Example 2.9.

3 Pushout of group schemes

In this section R is any complete discrete valuation ring with fraction and residue fields respectively denoted by K and k .

3.1 The finite case

Let $M = \text{Spec}(B)$ and $N = \text{Spec}(C)$ be finite and flat R -group schemes, i.e. B and C are free over R and finitely generated as R -modules. Let us assume that there is a K -group scheme morphism $\psi : M_K \rightarrow N_K$. An *upper bound* for M and N is a finite and flat R -group scheme U , provided with a model map $U \rightarrow M$ and a R -group scheme morphism $\varphi : U \rightarrow N$ which generically coincides with $\psi : M_K \rightarrow N_K$. A *lower bound* for M and N is a finite and flat R -group scheme L , provided with a model map $N \rightarrow L$ and a R -group scheme morphism $\delta : M \rightarrow L$ which generically coincides with $\psi : M_K \rightarrow N_K$. The construction of an upper bound is easy: it is sufficient to set U as the schematic closure of M_K in $M \times N$ through the canonical closed immersion

$M_K \hookrightarrow M_K \times N_K$ (and this holds when the base is any Dedekind scheme). Now consider the following commutative diagram

$$\begin{array}{ccc} & & M \\ & \nearrow m & \\ U & & \\ & \searrow n & \\ & & N \end{array} \quad (6)$$

where $U = \text{Spec}(A)$ is any upper bound. We are now going to study the existence of a pushout $M \sqcup_U N$ in the category of finite and flat R -group schemes. We prove the following

Lemma 3.1. *The pushout of (6) in the category of finite and flat R -group schemes exists. Moreover $M \sqcup_U N$ is a lower bound for M and N .*

Proof. Notation being as in the beginning of this section, we have the following diagram of commutative R -Hopf algebras:

$$\begin{array}{ccc} & A & \\ B & \nearrow & \nwarrow C \end{array}$$

which, dualizing, gives rise to the following diagram of cocommutative (but possibly non commutative) R -Hopf algebras:

$$\begin{array}{ccc} & A^\vee & \\ B^\vee & \nwarrow f & \searrow g C^\vee. \end{array}$$

Let us consider the cocommutative R -Hopf algebra $F(B^\vee *_{A^\vee} C^\vee)$ constructed in Corollary 2.15. Now we take the spectrum of its dual $P := \text{Spec}(F(B^\vee *_{A^\vee} C^\vee)^\vee)$. First of all we observe that $F(B^\vee *_{A^\vee} C^\vee)^\vee$ is commutative as $F(B^\vee *_{A^\vee} C^\vee)$ is cocommutative so that taking its spectrum does make sense. It remains to prove that the commutative diagram

$$\begin{array}{ccccc} & & M & & \\ & \nearrow m & & \searrow & \\ U & & & & P \\ & \searrow n & & \nearrow & \\ & & N & & \end{array}$$

is in fact a pushout in the category of finite and flat R -group schemes. But this follows from the fact that $F(B^\vee *_{A^\vee} C^\vee)$ is a pushout in $R\text{-}\mathcal{H}opf_{ff}$. That $M \sqcup_U N := P$ is a lower bound for M and N is also clear by construction. \square

Theorem 3.2. *The pushout of (6) in the category of affine R -group schemes exists.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
 & & M & & \\
 & \nearrow m & & \searrow u & \\
 U & & & & Q \\
 & \searrow n & & \nearrow v & \\
 & & N & &
 \end{array}$$

P

where P is the pushout of (6) in the category of finite and flat R -group schemes constructed in Lemma 3.1 and $Q = \text{Spec}(D)$ is any affine R -group scheme. We are going to show that P is also the pushout of (6) in the category of affine R -group schemes. Let us factor u through $M' := \text{Spec}(B')$ via the morphisms $u' : M \rightarrow M'$ and $i : M' \rightarrow Q$ where i is a closed immersion and u' is a schematically dominant morphism (i.e. the induced morphism $B' \rightarrow B$ is injective) so that M' is a finite and flat R -group scheme since M is. Likewise we factor v through the finite and flat R -group scheme $N' := \text{Spec}(C')$ via the schematically dominant morphism $v' : N \rightarrow N'$ and the closed immersion $j : N' \rightarrow Q$. Now consider the finite R -group scheme (it needs not be flat a priori) $M' \times_Q N'$ and the natural closed immersions $i' : M' \times_Q N' \hookrightarrow M'$ and $j' : M' \times_Q N' \hookrightarrow N'$. So let us denote by $r : U \rightarrow M' \times_Q N'$ the universal morphism, then we have the following commutative diagram

$$\begin{array}{ccccccc}
 & & M & \xrightarrow{u'} & M' & & \\
 & \nearrow m & & & \nearrow i' & & \\
 U & \xrightarrow{r} & M' \times_Q N' & & & & Q \\
 & \searrow n & & \searrow j' & & & \\
 & & N & \xrightarrow{v'} & N' & \xrightarrow{j} & \\
 & & & & & &
 \end{array}$$

and in particular we have the following commutative diagram of R -algebras

$$\begin{array}{ccc}
 A & \xleftarrow{\quad} & B' \\
 & \searrow & \downarrow \\
 & & B' \otimes_D C'
 \end{array}$$

where $B' \rightarrow B' \otimes_D C'$ is surjective but also injective since $B' \hookrightarrow A$ is (recall that $m : U \rightarrow M$ is a model map). Hence i' is an isomorphism so there exists a universal morphism $P \rightarrow N'$ by Lemma 3.1 and we conclude. \square

Remark 3.3. If U' is any other upper bound for M and N then there is a canonical isomorphism $M \sqcup_U N \simeq M \sqcup_{U'} N$: this is a consequence of Remark 2.12. However this does not mean that the lower bound for M and N is unique, which is clearly not true in general.

3.2 The quasi-finite case

Let $M = \text{Spec}(B)$ and $N = \text{Spec}(C)$ be quasi-finite (by this we will always mean affine and of finite type over R , with finite special and generic fibers) and flat R -group schemes. It is known (see [1], §7.3 p.179) that any quasi-finite R -group scheme H has a finite part H_f , that is a open and closed subscheme of H which consists of the special fibre H_k and of all points of the generic fibre which specialize to the special fibre. It is thus flat over R if H is.

Remark 3.4. If $H = \text{Spec}(A)$ is a quasi-finite and flat R -group scheme then its finite part coincides with $\text{Spec}(A^{\vee\vee})$ where $A^{\vee\vee}$ is the double dual of A : this follows from the fact that $A \simeq_{R\text{-mod}} K^{\oplus t} \oplus R^{\oplus s}$ (cf. Theorem 2.10) so $A^\vee \simeq_{R\text{-mod}} R^{\oplus s}$ is an R -Hopf algebra and not just an R -algebra. Hence the canonical surjection $A \twoheadrightarrow A^{\vee\vee}$ gives the desired closed immersion $H_f \hookrightarrow H$ of group schemes. However this fact will not be necessary in the remainder of this paper.

Let us assume that there is a K -group scheme morphism $\psi : M_K \rightarrow N_K$. We define upper and lower bounds exactly as in the finite case. One can easily construct an upper bound U for M and N simply proceeding as in §3.1. So U will be in general a quasi-finite and flat R -group scheme. For the lower bound it will be a little bit more complicated. So consider again the commutative diagram (6) where $U = \text{Spec}(A)$ is any upper bound. We are going to study the existence of a pushout $M \sqcup_U N$ in the category of affine R -group schemes. We prove the following

Theorem 3.5. *Assume that N_K admits a finite and flat model over R . Then the pushout of (6) in the category of affine R -group schemes exists. Moreover $M \sqcup_U N$ is a lower bound for M and N .*

Proof. Let N' denote a finite and flat R -model for N_K , i.e. a finite and flat R -group scheme whose generic fibre is isomorphic to N_K . Consider the finite part M_f and N_f of, respectively, M and N . Compose the closed immersion $M_{f,K} \rightarrow M_K$ with $\psi_K : M_K \rightarrow N_K$ thus obtaining a morphism $M_{f,K} \rightarrow N_K$. By Theorem 3.2 we construct a lower bound L_1 for M_f and N' , which is finite and flat over R , generically isomorphic to N_K , then it is already a lower bound for M and N' . Considering the closed immersion $N_{f,K} \rightarrow N_K$ we also construct a lower bound L_2 for N_f and N' , which is finite and flat over R , generically isomorphic to N_K , then it is already a lower bound for N and N' . So a lower

bound L for L_1 and L_2 (which are generically isomorphic) exists by Theorem 3.2 and is also a lower bound for M and N . We still need to compute the pushout of M and N over U : let us set $U_f := \text{Spec}(A_f)$, $M_f := \text{Spec}(B_f)$, $N_f := \text{Spec}(C_f)$ and $L := \text{Spec}(D)$. Consider the natural R -bialgebra morphism (cf. Proposition 2.13) $B_f^\vee *_{A_f^\vee} C_f^\vee \rightarrow D^\vee$ and factor it as follows

$$B_f^\vee *_{A_f^\vee} C_f^\vee \twoheadrightarrow E^\subset \longrightarrow D^\vee$$

where E is a cocommutative R -bialgebra which is flat and finitely generated as an R -module because D^\vee is. Consider the morphism $S_{B_f^\vee *_{A_f^\vee} C_f^\vee} : B_f^\vee *_{A_f^\vee} C_f^\vee \rightarrow B_f^\vee *_{A_f^\vee} C_f^\vee$ constructed in Remark 2.14; the commutative diagram

$$\begin{array}{ccccc} B_f^\vee *_{A_f^\vee} C_f^\vee & \twoheadrightarrow & E^\subset & \longrightarrow & D^\vee \\ S_{B_f^\vee *_{A_f^\vee} C_f^\vee} \downarrow & & & & \downarrow S_{D^\vee} \\ B_f^\vee *_{A_f^\vee} C_f^\vee & \twoheadrightarrow & E^\subset & \longrightarrow & D^\vee \end{array}$$

induces a anti-morphism of R -algebras

$$S_E : E \rightarrow E$$

which gives E a natural structure of R -Hopf algebra: indeed $m_E \circ (id_E \otimes S_E) \circ m_E = u_E \circ \varepsilon_E$ and $m_E \circ (S_E \otimes id_E) \circ m_E = u_E \circ \varepsilon_E$ since the same equalities hold for D^\vee . It is now sufficient to take the union of $\text{Spec}(E^\vee)$ and $N_K \simeq L_K$ in order to construct a quasi-finite and flat R -group scheme P which is certainly a pushout in the category of quasi-finite and flat R -group schemes. Arguing as in the proof of Theorem 3.2 we can deduce that P is also a pushout in the category of affine R -group schemes. \square

Remark 3.6. As in Remark 3.3 one observes that if U' is any other upper bound for M and N then there is a canonical isomorphism $M \sqcup_U N \simeq M \sqcup_{U'} N$: indeed E , as constructed in the proof, is the only quotient of $B_f^\vee *_{A_f^\vee} C_f^\vee$, R -flat which over K gives $B_{fK}^\vee *_{A_{fK}^\vee} C_{fK}^\vee \rightarrow D_{fK}^\vee$ and this does not depend on A_f . The same will hold for Corollary 3.7 and will be used in Corollary 3.9.

Corollary 3.7. *When N_K is étale then after possibly a finite extension of scalars the pushout of (6) in the category of affine R -group schemes exists. Again $M \sqcup_U N$ is a lower bound for M and N .*

Proof. Clear since after possibly a finite extension K' of K the K -group scheme N_K becomes constant then it certainly admits a finite, constant (so flat) model over R' , the integral closure of R in K' . \square

Let K' be a finite extension of K and R' the integral closure of R in K' then R' is a complete discrete valuation ring. Assume that W is a torsion-free R module of finite rank n then we have the following

Lemma 3.8. *If $W \otimes_R R'$ is finitely generated as an R' -module then W is finitely generated as an R -module too (thus free).*

Proof. By Theorem 2.10 $W \simeq_{R\text{-mod}} K^{\oplus n-s} \oplus R^{\oplus s}$, where $s = \dim_k(W_k)$, hence $W \otimes_R R' \simeq_{R'\text{-mod}} K'^{\oplus n-s} \oplus R'^{\oplus s}$ so if $W \otimes_R R'$ is finitely generated as an R' -module then $n - s = 0$ and we conclude. \square

This will be used in the following

Corollary 3.9. *When R is complete and N_K is étale then the pushout of (6) in the category of affine R -group schemes exists. Again $M \sqcup_U N$ is a lower bound for M and N .*

Proof. Again $U_f = \text{Spec}(A_f)$, $M_f = \text{Spec}(B_f)$, $N_f = \text{Spec}(C_f)$ will denote the finite part of U , M and N respectively. Let us consider the duals $A_f^\vee = A^{\vee\vee}$, $B_f^\vee = B^{\vee\vee}$ and $C_f^\vee = C^{\vee\vee}$ and the commutative diagram

$$\begin{array}{ccccc}
 & A_{fK}^\vee & \xleftarrow{\quad} & A_f^\vee & \\
 & \swarrow & & \searrow & \\
 B_{fK}^\vee & \xleftarrow{\quad} & C_{fK}^\vee & \xleftarrow{\quad} & B_f^\vee & \xleftarrow{\quad} & C_f^\vee \\
 & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow \\
 & B_{fK}^\vee *_{A_{fK}^\vee} C_{fK}^\vee & \xleftarrow{\quad} & B_f^\vee *_{A_f^\vee} C_f^\vee & \\
 & \downarrow & & & \\
 & E & & & \\
 & \downarrow & & & \\
 & C^\vee & & &
 \end{array} \tag{7}$$

where E comes from the factorisation of the universal morphism $B_{fK}^\vee *_{A_{fK}^\vee} C_{fK}^\vee \rightarrow C^\vee$. Arguing as in Theorem 3.5 we provide it with a natural structure of K -Hopf algebra. Using again [5], Lemme (2.8.1.1) we construct the unique quotient

$$B_f^\vee *_{A_f^\vee} C_f^\vee \twoheadrightarrow E'$$

which is R -flat and which generically gives

$$B_{fK}^\vee *_{A_{fK}^\vee} C_{fK}^\vee \twoheadrightarrow E.$$

Thus E' has naturally a structure of a cocommutative R -Hopf algebra: indeed it inherits from $B_f^\vee *_{A_f^\vee} C_f^\vee$ a cocommutative R -coalgebra structure and by means of [5], (2.8.3) an anti-morphism of R -algebras $S'_E : E' \rightarrow E'$ which is a coinverse since tensoring it over K we obtain $S_E : E \rightarrow E$ which is a coinverse for E . If we prove that E' is finitely generated as a R -module then $\text{Spec}(E'^\vee)$ glued to N

is the desired pushout. So now it remains to prove that E' is finitely generated as a R -module: let $K \rightarrow K'$ be a finite field extension such that $N_{K'}$ admits a finite and flat model over R' , the integral closure of R in K' . Then by Corollary 3.7 and Remark 3.6 $E' \otimes_R R'$ is R -finite and flat. Lemma 3.8 implies that E' is R -finite and flat too. \square

Remark 3.10. It is less elegant but still true that Corollary 3.9 holds for all those N_K that admits a finite and flat R -model after possibly a finite extension of scalars and étale ones are just a particular case. Observe furthermore that, following the proof, in the situation of both Theorem 3.5 and Corollary 3.7 one can find a finite and flat lower bound for M and N . This can be false in the situation of Corollary 3.9.

3.3 Cokernels and quotients

In a category \mathcal{C} with zero object $0_{\mathcal{C}}$ (that is an object which is both initial and final), we can define the cokernel of a morphism $f : A \rightarrow B$ (see for instance [8], III, §3) which turns out to be the pushout $0_{\mathcal{C}} \sqcup_A B$ of the obvious diagram. As explained in the introduction in this section we are going to describe, in Proposition 3.12, a new and easy proof for a well known result. First we need a lemma:

Lemma 3.11. *Let R be a Dedekind ring or a field, A , B and C R -Hopf algebras provided with R -Hopf algebra morphisms $f : A \rightarrow B$ and $g : A \rightarrow C$. Then the star product $B \star_A C$ defined in Definition 2.3 has a natural structure of R -Hopf algebra.*

Proof. First we prove the existence of the comultiplication $\Delta_{B \star_A C}$: it is sufficient to consider the following diagram

$$\begin{array}{ccccc}
 & B & \xrightarrow{\Delta_B} & B \otimes_R B & \\
 f \nearrow & & \searrow u & \nearrow f \otimes f & \searrow u \otimes u \\
 A & \xrightarrow{\Delta_A} & B \star_A C & \xrightarrow{\quad} & (B \star_A C) \otimes_R (B \star_A C) \\
 g \searrow & \nearrow v & \xrightarrow{\quad} & \nearrow g \otimes g & \searrow v \otimes v \\
 & C & \xrightarrow{\Delta_C} & C \otimes_R C &
 \end{array}$$

$\Delta_{B \star_A C}$ is indicated by a dashed arrow from $B \star_A C$ to $(B \star_A C) \otimes_R (B \star_A C)$.

where the existence of $\Delta_{B \star_A C}$ is ensured by Proposition 2.4. The existence of $\varepsilon_{B \star_A C}$ is easier and an argument similar to the one used in Remark 2.14 ensures the existence of an anti-morphism $S_{B \star_A C} : B \star_A C \rightarrow B \star_A C$ which is compatible with $S_{B \otimes_R C}$, i.e. if $\lambda : B \otimes_R C \rightarrow B \star_A C$ denotes the canonical projection then $\lambda \circ S_{B \otimes_R C} = S_{B \star_A C} \circ \lambda$. From this we deduce that $S_{B \star_A C}$ is the desired coinverse for $B \star_A C$. \square

Proposition 3.12. *Let R be a Dedekind ring or a field, G and H two finite and flat R -group schemes and $f : H \rightarrow G$ a morphism of R -group schemes. Then the cokernel of f exists in the category of R -affine group schemes.*

Proof. The zero object in the category of R -affine group schemes is $\text{Spec}(R)$. Let us set $H = \text{Spec}(A)$ and $G = \text{Spec}(B)$. Then we first compute the pushout in the category of R -Hopf algebras of the diagram

$$\begin{array}{ccc} & A^\vee & \\ \swarrow & & \searrow \\ B^\vee & & R. \end{array} \quad (8)$$

In Example 2.8 we have observed that $W := B^\vee *_{A^\vee} R \simeq B^\vee \star_{A^\vee} R$ canonically. That W has a natural structure of R -Hopf algebra follows from Lemma 3.11. If R is a Dedekind ring and W is not flat then we consider $F(W)$ (cf. Notation 2.7) which is flat and finitely generated and inherits the R -Hopf algebra structure. Since the case of a field is similar let us just consider the case of a Dedekind ring R : we are now going to prove that $\text{Spec}(F(W)^\vee)$ is the desired pushout. So let M be any affine R -group scheme, $v : G \rightarrow M$ an R -group scheme morphism and $u : \text{Spec}(R) \rightarrow M$ the natural inclusion (the unity map). Let us assume we have a commutative diagram

$$\begin{array}{ccc} & \text{Spec}(R) & \\ \nearrow & & \searrow u \\ H & & M \\ \searrow f & & \nearrow v \\ & G & \end{array} \quad (9)$$

Observe that we can assume M to be finite and flat, for if it is not we can factor v through a finite and flat (since G is) R -group scheme that makes a diagram similar to (9) commute. When M is finite and flat it is easy to construct a universal morphism $\text{Spec}(F(W)^\vee) \rightarrow M$ since $F(W)$ is easily seen to be the pushout of diagram (8) in $R\text{-Hopf}_{ff}$. \square

Corollary 3.13. *Let R be a Dedekind ring, G and H two finite and flat R -group schemes with H a closed and normal R -subgroup scheme of G . Then the quotient G/H exists in the category of R -affine group schemes.*

Proof. This follows directly from Proposition 3.12 where we take for $f : H \rightarrow G$ the given closed immersion. \square

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Marco Antei
Department of Mathematics
Ben Gurion University of the Negev
Be'er Sheva 84105, Israel
E-mail: anteim@math.bgu.ac.il
marco.antei@gmail.com